

Integral Geometry on the Lobachevsky Plane and the Conformal Wess-Zumino-Witten Model of Strings on an ADS_3 Background

Bogdan G. Dimitrov

Bogoliubov Laboratory of Theoretical Physics

Joint Institute for Nuclear Research, Dubna 141 980, Russia

email: bogdan@thsun1.jinr.ru

Abstract

The main purpose of the report is to provide some argumentation that three seemingly distinct approaches of 1. Giveon, Kutasov and Seiberg (hep-th/9806194); 2. Hemming, Keski-Vakkuri (hep-th/0110252); Maldacena, Ooguri (hep-th/0001053) and 3. I. Bars (hep-th/9503205) can be investigated by applying the mathematical methods of integral geometry on the Lobachevsky plane, developed previously by Gel'fand, Graev and Vilenkin. All these methods can be used for finding the transformations, leaving the Kac-Moody and Virasoro algebras invariant. The near-distance limit of the Conformal Field Theory of the $SL(2, R)$ WZW model of strings on an ADS_3 background can also be interpreted in terms of the Lobachevsky Geometry : the non - euclidean distance is conserved and the Lobachevsky formulae for the angle of parallelism is recovered. Some preliminary technique from integral geometry for inverting the modified integral representation for the Kac- Moody algebra has been demonstrated.

Keywords: string theory, conformal field theory, algebraic and integral geometry, Anti de-Sitter space

1 Introduction and Statement of the Problem

The present paper has the purpose to illustrate the importance of the ideas and constructions of the Non-Euclidean (Lobachevsky) Geometry, which can be applied even today for solving some conceptually important problems in **string theory on an (Anti - DeSitter) ADS_3 background**. Presently one of the most widely discussed topics is the ADS/CFT correspondence, where the $3D$ "**Anti-DeSitter**" space is in fact the $3D$ **Lobachevsky space of constant negative curvature**, proposed first by Lobachevsky, Boyai (subsequently by Gauss, Beltrami and many others) yet in the 18th century. In this aspect one should mention also the lectures of N.A. Chernikov [1], which review and clarify such fundamental notions as the horosphere, the horocycles, the Poincare model and etc., constituting the basic necessary knowledge for application of the integro-geometric approach of Gel'fand, Graev and Vilenkin [2, 3].

In the last several years there has been a considerable progress in constructing **Conformal Field Theories (CFT)** on an **Riemann surface** [4, 5]. One of the main successes of this theory is that on the base of the complex automorphisms of the Riemann sphere (isomorphic to the group $SL(2, C)/Z_2$ and represented as a composition of **translation, the scaling transformation and the special conformal transformation**) and by means of the duality theorem for the (specially constructed) **vertex operator V** , it turns out to be possible to derive the **Virasoro algebra**

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \quad (1)$$

without the use of the OPE relations

$$T(z)T(v) = \frac{c}{2(z-v)^4} + \frac{2}{(z-v)^2}T(v) + \frac{1}{z-v}T'(v) + \quad , \quad (2)$$

$$T(z)J(v) = \frac{J^a(v)}{(z-v)^2} + \frac{1}{(z-v)}J^{a'}(v) + \quad , \quad (3)$$

$$J^a(z)J^b(v) = \frac{k\delta^{ab}}{(z-v)^2} + \frac{f^{abc}}{(z-v)}J^c(v) + \quad (4)$$

for the stress tensor $T(z)$ and for the conformal current $J^a(z)$ - unlike the approach in the first papers on CFT [6, 7, 8]. Three major developmens in the contemporary theory will be mentioned, which may seem to be distinct, but in fact the purpose here will be to show that an unifying mathematical approach can be proposed in respect to all of them.

The first development is related to the problem about the **spectral flow** [9, 10, 11] , which generates new classical solutions in the WZW model after applying the operation

$$g_+ = exp \left[\frac{i}{2} \omega_R x^+ \sigma_2 \right] \tilde{g}_+(x^+) \quad ; \quad g_- = \tilde{g}_-(x^-) exp \left[\frac{i}{2} \omega_L x^- \sigma_2 \right] \quad , \quad (5)$$

where the group element g factorizes into the product of the right and left group elements g_+ and g_- , $x^+ = \tau + \sigma$ and $x^- = \tau - \sigma$ are the (right and left) coordinates on the worldsheet and ω_L, ω_R . are the eigenvalues of the left and right conformal currents. For the case of the BTZ black hole [11], the spectral flow **defines a mapping between the string worldsheet coordinates τ, σ and the global spacetime coordinates $r, \hat{t}, \hat{\Phi}$**

$$\hat{t} \rightarrow t + \frac{1}{2} [(\omega_+ + \omega_-)\tau + (\omega_+ - \omega_-)\sigma] \quad (6)$$

$$\hat{\Phi} \rightarrow \Phi + \frac{1}{2} [(\omega_+ + \omega_-)\sigma + (\omega_+ - \omega_-)\tau] \quad , \quad (7)$$

under which the components of the conformal currents and of the Virasoro generators should transform as

$$\tilde{J}_R^2 = J_R^2 + \frac{k}{2} w_+ \quad ; \quad \tilde{J}_R^\pm = J_R^\pm exp(\mp w_+ x^+) \quad (8)$$

$$\tilde{L}_n = L_n + w_+ J_n^2 + \frac{k}{4} w_+^2 \delta_{n,0} \quad , \quad (9)$$

so that the invariance of the Kac - Moody and Virasoro algebras is ensured.

The second development is proposed in a paper of Giveon, Kutasov and Seiberg [12]. Let Q_0^a are the **spacetime conserved charges**

$$Q_0^a \equiv \oint dz J^a(z) \quad , \quad (10)$$

which evidently satisfy the closed Lie algebra $[Q_0^a, Q_0^b] = if_c^{ab} Q_0^c$ and the conformal currents $J^a(z)$ ($a = 3, +, -$ for the $SL(2, R)$ case) satisfy the OPE relations (4), where k is the level of the string worldsheet algebra. In order to **extend the zero - mode symmetry of the space time charges Q_0^a to an infinite symmetry of the charges Q_n^a** , satisfying the affine Lie algebra

$$[Q_n^a, Q_m^b] = if_c^{ab} Q_{n+m}^c + \frac{\tilde{k}}{2} n \delta^{ab} \delta_{n+m,0} \quad , \quad (11)$$

the authors propose to define Q_n^a by multiplying the integrand $J^a(z)$ in (10) with a holomorphic function $\gamma^n(z)$, i. e $Q_n^a \equiv \oint dz J^a(z) \gamma^n(z)$. It is important that the function $\gamma^n(z)$ comes from the "microscopic" definition of a **vertex operator [13] in string theory** $V_{j\overline{m}} = \gamma^{j+m} \gamma^{-j+\overline{m}} \exp(\frac{2j}{\alpha} \Phi)$, this time for values of the quantum numbers $j = 0, \overline{m} = 0, m = n$. Then the "modified" charge operators Q_n^a will satisfy the affine Lie algebra (11) only if the **level \tilde{k} of the affine Lie algebra in spacetime** is p -times the **level k (see eq. 4) of the string worldsheet algebra**, i.e. $\tilde{k} = pk$. The number $p \equiv \oint dz \frac{\partial_z \gamma}{\gamma}$ "counts" the number of times the string worldsheet wraps around the angular coordinate Θ .

Now what is the conclusion from all these reasonings? The function $\gamma^n(z)$ in the "microscopic" definition of the vertex operator also enters in the expression for the metric

$$ds^2 = L^2 (d\Phi^2 + e^{2\Phi} d\gamma d\overline{\gamma}) \quad , \quad (12)$$

obtained after a given parametrization of the global ADS_3 (Lobachevsky) coordinates (X_0, X_1, X_2, X_3) , defined on the hyperboloid $X_0^2 - X_1^2 - X_2^2 - X_3^2 = L^2$. Furthermore, the conformal currents in the WZW model

$$J_L^a = k Tr(T^a g^{-1} \partial_z g) \quad \quad J_R^a = k Tr(T^a \partial_z g \cdot g^{-1}) \quad , \quad (13)$$

in which the group element g is parametrized by means also of the global coordinates (X_0, X_1, X_2, X_3) (see for some details [14]), are identified with the currents in the OPE relations and the Kac - Moody and Virasoro algebras in the WZW model. **Therefore, we come to the following important problem: How are the global symmetries of the $ADS3$ (Lobachevsky) spacetime related to the transformations, which leave the Kac - Moody and Virasoro algebras invariant?**

To confirm the importance of the stated problem, let us mention briefly the **third development** in the string theory on an $ADS3$ spacetime in some of the papers of I. Bars [15]. He assumes that there is **an additional zero mode** (proportional to lnz -pieces) that is present in the local conserved $SL(2, R)$ currents of the WZW model. Similarly to the previous developments, in [15] and some other papers, the transformations of the conformal currents (and of the stress tensor), leaving the Kac-Moody and Virasoro algebras invariant are also found.

2 Basic Assumptions, Objectives of the Present Research and Results, Related to Lobachevsky Geometry

The present paper and the performed research as a whole have the following objectives:

1. To clarify the non-euclidean geometrical meaning of the found in [14] algebraic relation in CP^3 from the OPE relations with the conformal current (only). A central result will be that the *non-euclidean distance* ρ , defined by means of the **anharmonic relation**

$$\rho = r \ln |(w \ z \ v \ t)| = r \ln \left| \frac{(w-v)(z-t)}{(z-v)(w-t)} \right| \quad (19)$$

for four points w, z, v and t on the Lobachevsky (hyperbolic) plane (r is a scale factor), is a **constant** (complex number). Since no OPE relations with the stress tensor have been used and therefore there is **no translation, this might be interpreted as being physically consistent with the initial assumption about rotations (only)**.

2. After an identification of the conformal currents in the OPE relations with the conformal current in the WZW , to investigate the $SL(2, R)$ OPE relations. As **one condition** (several others have been found, but represented by much more complicated expressions) for **consistency of the OPE relations** and after some redefinitions of the parametrization variables (Ψ, Θ, Φ) , the well-known **Lobachevsky formulae [1, 16] for the angle of parallelism** will be recovered:

$$\Pi(\rho) = 2 \arctg(\exp(-\frac{\rho}{R})) \quad , \quad (20)$$

where ρ is the noneuclidean distance (19) for the worldsheet manifold and the number R is the so called "curvature radius" of the Lobachevsky plane - a measure for the "noneuclideanity" of the Lobachevsky geometry.

Therefore, the applicability of Lobachevsky geometry for treating the $SL(2, R)$ OPE relations has been confirmed **twice**: before and after the identification of the conformal currents. This can hardly be a simple coincidence of calculations.

In both cases only the pole terms in the OPE relations have been taken into account and all the s. c. "regular" terms have been discarded. Consequently, this is the "near - distance" approximation limit of the points on the complex plane, which is in accord with taking into account only the first two terms in the variation $\delta_\varepsilon A_j(z) = \sum_{k=0}^{\nu_j} B_j^{(k-1)}(z) \frac{d^k}{dz^k} \varepsilon(z)$ (ν_j - integer numbers) of the local field $A_j(z)$ [5]. This comes from the requirements $\Delta_j \geq 0$ (Δ_j - conformal dimension of the local fields $A_j(z)$; $\Delta_{j,(k-1)}$ - of the fields $B_j^{(k-1)}(z)$) $\Delta_{j,(k-1)} = \Delta_j + 1 - k$ and $\Delta_{j,(k-1)} = 0$ (the lowest possible value), which guarantee that any correlators in the CFT would be convergent with the increase of distance (see [6]).

3. In order to incorporate the symmetries of the global ADS coordinates in the

problem about the transformations, leaving the Kac- Moody and Virasoro algebras invariant, and with the purpose to unite the three distinct approaches, discussed in the Introduction, in [14] it was proposed to **modify the integral definition** of the Virasoro and of the conformal current generators (here only the formulae for the Virasoro generators will be written)

$$\widehat{L}_n(\eta)A_j(z, \bar{z}) = \oint_C \left[\oint_{C_1} T(\zeta)(\zeta - z)^{n+1} \delta([\widetilde{X}, \eta] + 1) A_j(z, \bar{z}) d\widetilde{X} \right] d\zeta \quad , \quad (21)$$

where (in the spirit of the integro - geometric approach of Gel'fand, Graev and Vilenkin [2, 3]) an additional integration is introduced on the *ADS* hyperboloid coordinates \widetilde{X} ($d\widetilde{X} \equiv d\widetilde{X}^0 d\widetilde{X}^1 d\widetilde{X}^2 d\widetilde{X}^3$) and the presence of the delta - function signifies that the integration is on the **horosphere**

$$[\widetilde{X}, \eta] + 1 = -X^0\eta^0 + X^3\eta^3 + X^1\eta^1 + X^2\eta^2 + 1 = 0 \quad , \quad (22)$$

intersecting the cone $[\eta, \eta] = 0$. **By definition, the horosphere in Lobachevsky geometry is the surface, orthogonal to the set of parallel lines, passing through one and the same point on the absolute [1, 16], and the absolute by itself is simply the set of "points" (a "point" may be a straight line, or a hyperplane) at infinity. The most essential property of the horosphere is that on it the usual Euclidean geometry is realized.** That is why the "harmonic" measure, defined by $\delta([\widetilde{X}, \eta] + 1) d\widetilde{X}$ is invariant under rotations around the \widetilde{X} -point.

The most important result in [14] was that the Virasoro algebra $[L_n, J_m^a] = -mJ_{n+m}^a$ in terms of the newly defined operators can be written as

$$\oint_{(\zeta_1, X_1)} F(\zeta_1, \zeta_2, z) \delta([X_1, \eta_1] + 1) d\zeta_1 d\widetilde{X} = -mJ^a(\zeta_2)(\zeta_2 - z)^{m+n} \quad , \quad (23)$$

where $F(\zeta_1, \zeta_2, z) \equiv T(\zeta_1)J^a(\zeta_2) [(\zeta_1 - z)^{n+1}(\zeta_2 - z)^m t_j^m - (\zeta_1 - z)^m(\zeta_2 - z)^{n+1} t_j^n]$. Now one way of realizing the large - scale symmetries is when the integration over the $d\widetilde{X} \equiv d\widetilde{X}^0 d\widetilde{X}^1 d\widetilde{X}^2 d\widetilde{X}^3$ variables is performed, making the variable change

$q = [\tilde{X}, \eta] + 1$, integrating over $\int \frac{\partial X^0}{\partial q} \delta(q) dq d\tilde{X}^1 d\tilde{X}^2 d\tilde{X}^3$ and lastly, taking into account the hyperboloid equation $-(\tilde{X}^0)^2 - (\tilde{X}^1)^2 + (\tilde{X}^2)^2 + (\tilde{X}^3)^2 = 1$. This trivial calculation shall not be performed here. The details, including an extensive review of the approaches in integral geometry and of the basic notions in Lobachevsky geometry, will be given in [17].

Of course, if one knows the algebraic surface $p(\zeta_1, \zeta_2, z)$, on which these three complex variables are defined, then in the integral (23) one may introduce an integration over the delta function $\delta(p(\zeta_1, \zeta_2, z))$, which will allow to perform the integration over the ζ_1 - variable in (23). Unfortunately, as we shall later show (and was shown in [14]), such a relation will be obtained from the conformal OPE relations with the conformal current **only, so one does not have the right to apply it to the** Virasoro algebra $[L_n, J_m^a] = -m J_{n+m}^a$, where the Virasoro generator L_n in (21) is related also to the stress tensor $T(\zeta)$. In principle, the problem about deriving an algebraic relation from all the OPE relations remains unsolved, and it requires significant efforts, since, as it is well known, the transformation properties of the stress tensor are nontrivial - it is transformed by means of the s. c. **Schwarzian**. Still, another approach from integral geometry - the **Radon's transformation in complex space** can be used, and its inverse one is well-known [2].

Further, it shall be proposed to take advantage of knowing the algebraic relation

$$p(\zeta_1, \zeta_2, z) \equiv \frac{1}{(\zeta_1 - z)^2} + \frac{1}{(\zeta_2 - z)^2} = 0 \quad (24)$$

for the complex points on the worldsheet, on which the integral operators in the **Kac - Moody algebra**

$$[J_n^a, J_m^b] = i f_c^{ab} J_{n+m}^c + \frac{k}{2} n \delta^{ab} \delta_{n+m,0} \quad (25)$$

is defined. This time integrals of the type [2, 3]

$$h(\eta) = \int f(X) \delta([X, \eta] - 1) dX \quad (26)$$

are to be "inverted". The integral is defined on the upper-half $[X, X] > 0$, where all "lines" and "hyperplanes" remain inside the absolute $[X, X] = 0$. The inversion

formulae for the the function $f(X)$, taken at the point $X = a$, is [2, 3]:

$$f(a) = -\frac{1}{8\pi^2} \int \delta''([a, \eta] - 1) h(\eta) d\eta \quad . \quad (27)$$

For the opposite case $[X, X] < 0$, when the so called "imaginary Lobachevsky space" (on which the distance can be either a real number from 1 to ∞ , or an imaginary number), the inversion formulae is more complicated, and shall not be considered here.

3 Conservation of Non-Euclidean Distance from the OPE Relations for the Conformal Current

The conservation of non-euclidean distance will be deduced from the already found in [14] algebraic relation (24) $\frac{1}{(z-v)^2} + \frac{1}{(w-v)^2} = 0$ for the points z, v and w . Let us remind briefly that it was obtained from (4), written for $(A, B) = (3, +), (3, -), (+, -), (+, +), (-, -), (3, 3)$, multiplying (4) consequently by $J^C(v)$ and again applying the obtained OPE relations for the $SL(2, R)$ case.

Making use of the definition (19) for non-euclidean distance, it is easily seen that the above algebraic relation can be rewritten as an **anharmonic relation for the four points w, z, v and $t = \infty$** :

$$\varepsilon i = \frac{w - v}{z - v} = (w \ z \ v \ \infty) \quad (\varepsilon = +1 \text{ or } \varepsilon = -1) \quad . \quad (28)$$

The conservation of the non-euclidean distance immediately follows

$$\tilde{\rho} = r \ln |(w \ z \ v \ \infty)| = r \frac{i\pi}{2} \text{ if } \varepsilon = 1 \text{ and } = r \frac{3\pi i}{2} \text{ if } \varepsilon = -1 \quad (29)$$

and therefore it determines an **equidistant surface in CP^3** . From hyperbolic geometry it is known that a **mapping of points is a Mobius one if and only if the anharmonic relation is conserved**. But since Mobius invariance is the basic assumption in the OPE relations, we receive a consistent result, but this time from an algebraic point of view!

The point at infinity can be chosen also at $z = \infty$ or at $w = \infty$. Then, writing down the algebraic relation for the points v, w and t and combining the two relations, the conservation of the sum of two noneuclidean distances can be obtained:

$$\ln |(t \infty w v)| + \ln |(t z v \infty)| = 0 \text{ if } \varepsilon = 1 \text{ and } = r \exp\left(\frac{i\pi}{2}\right) \text{ if } \varepsilon = -1. \quad (30)$$

However, in terms of the real variables $x_1, y_1, x_2, y_2, x_3, y_3$, where $z = x_1 + iy_1$, $v = x_2 + iy_2$ and $w = x_3 + iy_3$, the conservation of the non-euclidean distance is not so obvious. After solving the equations for the real and the imaginary parts of the algebraic relation, one gets

$$\frac{x_1 - x_2}{x_3 - x_2} \cdot \frac{y_1 - y_2}{y_3 - y_2} = -1, \quad (31)$$

which can be represented also as

$$-1 = (x_1 \ x_3 \ x_2 \ y_2) \cdot (y_1 \ y_3 \ y_2 \ x_2) \cdot F; \text{ with } F \equiv \frac{(x_1 - y_2)(y_1 - x_2)}{(x_3 - y_2)(y_3 - x_2)}. \quad (32)$$

Consequently, if $x_2 \iff y_2$ and $\tilde{v} \equiv i\bar{v}$ satisfies the original algebraic relation (note that then $F = -1$, i. e. F is the original expression (31)), then a conservation of the non - euclidean distance will also follow:

$$\ln |(x_1 \ x_3 \ x_2 \ y_2)| = \ln |(y_1 \ y_3 \ x_2 \ y_2)|. \quad (33)$$

This relation in fact means that the non-euclidean distance is conserved in respect to the symmetry changes $x_1 \iff y_1$ and $x_3 \iff y_3$. In the general case, however, **the symmetries of the algebraic relation and of the expression for the noneuclidean distance are different.**

4 Recovering the Lobachevsky Formulae for the Angle of Parallelism from the OPE Relations

(4)

As already mentioned, the conformal currents in the WZW model (where the group element is parametrized by the ADS global coordinates) will be identified with the currents in the OPE relations. The following additional and very simple assumptions will hold:

1. **The ADS global coordinates X^μ depend on the "angular" type coordinates (Φ, Ψ, Θ) , defined on the complex coordinates z and \bar{z} of the string world-sheet**

$$X^\mu \equiv X^\mu(\Phi(z, \bar{z}), \Psi(z, \bar{z}), \Theta(z, \bar{z})) \quad , \quad (34)$$

where $\mu = 0, 1, 2, 3$, and both the ADS coordinates X^μ and the variables (Φ, Ψ, Θ) are invariant under complex reparametrizations:

$$X^\mu(z) \equiv X^\mu(w(z)) \quad ; \quad (\Phi(w(z)), \Psi(w(z)), \Theta(w(z))) \equiv (\Phi(z), \Psi(z), \Theta(z)) \quad . \quad (35)$$

From here by simple differentiation it can be obtained that

$$\frac{\partial \Phi(z)}{\partial z} - \frac{\partial \Phi(z)}{\partial w} \frac{\partial w}{\partial z} = 0 \quad (36)$$

(and of course, analogously for the other two derivatives $\frac{\partial \Psi(z)}{\partial z}$ and $\frac{\partial \Theta(z)}{\partial z}$), but with the important notice - only if the derivatives $\frac{\partial X^\mu(z)}{\partial \Phi}$, $\frac{\partial X^\mu(z)}{\partial \Psi}$ and $\frac{\partial X^\mu(z)}{\partial \Theta}$ are **arbitrary**. It will be proved in the paper [17], however, that if one specifies the metric, for example in the form

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 \quad , \quad (37)$$

then the derivatives of the global ADS coordinates X^μ for $\mu = 3$ are no longer independent.

2. The derivatives $\frac{\partial\Phi(z)}{\partial z}$, $\frac{\partial\Psi(z)}{\partial z}$ and $\frac{\partial\Theta(z)}{\partial z}$ are found by making use of the found algebraic relation (24), but at the point $v = 0$, which means that the point w is obtained from the point z by **an orbifold rotation at an angle $\frac{\pi}{2}$, i. e.** $w = \varepsilon e^{\frac{i\pi}{2}} z$ ($\varepsilon = \pm 1$). Consequently the conformal generators $J_R^3(w)$, $J_R^+(w)$ and $J_R^-(w)$ are obtained to be also orbifold rotated around the point z , i.e. $J_R^3(w) = -i\varepsilon J_R^3(z)$; $J_R^+(w) = -i\varepsilon J_R^+(z)$ and $J_R^-(w) = -i\varepsilon J_R^-(z)$.

Further from the system of equations (4) for the conformal currents in the OPE relations for the $SL(2, R)$ case, one can obtain the following system of three first - order nonlinear differential equations for the variables Φ , Ψ and Θ :

$$\frac{\partial\Theta}{\partial z} = T(\Theta, \Phi, \Psi) ; \quad \frac{\partial\Phi}{\partial z} = Q(\Theta, \Phi, \Psi)T(\Theta, \Phi, \Psi) \quad ; \quad (38)$$

$$\frac{\partial\Psi}{\partial z} = P(\Theta, \Phi, \Psi)T(\Theta, \Phi, \Psi) \quad . \quad (39)$$

Each function Φ , Ψ and Θ is assumed to have a real and an imaginary part ($\Phi(z) \equiv \Phi_1(z) + i\Phi_2(z)$ and etc.), so after separating the real and the imaginary parts in each of the equations and after combining (but not solving) the obtained equations, one can derive the following simple consistency condition :

$$\tanh 2\Phi_1 = \pm \frac{k}{L^2} \quad . \quad (40)$$

where L is the DeSitter radius.

Now let $\Pi(\rho)$ is the **angle of parallelism in Lobachevsky geometry**, which represents the angle between the perpendicular (through a given point) towards a given line l_1 and the the parallel to l_1 line, drawn again through this point. After making the identifications

$$\Phi_1 \equiv \frac{\Pi(\rho)}{4} \quad ; \quad \exp(-\frac{\rho}{r}) = -i\varepsilon \frac{k}{L^2} \quad , \quad (41)$$

the **Lobachevsky formulae for the angle of parallelism** is recovered:

$$\Pi(\rho) = 2\arctg(\exp(-\frac{\rho}{r})) \quad , \quad (42)$$

where ρ is the hyperbolic distance $\rho = 2 \int_0^r \frac{dx}{1-x^2} = 2 \operatorname{arctanh} r$ (expressible also through the anharmonic relation), and the simple formulae $\tanh(ix) = itgx$ has been used.

5 Inversion Formulae from Integral Geometry - Application to the Kac- Moody Algebra

The main purpose in this section will be to **apply the inversion formulae (27) to the integral expression, which will be obtained below from the Kac-Moody algebra (25)**. In fact, this is the first step towards constructing the integro - geometric approach for treating simultaneously all the Kac - Moody and Virasoro algebras, based on applying various methods from integral geometry. Some details will also be given in [17].

The central idea here will be to reduce the double integration (in the L. H. S.) in the integral representation of the commutator $[J_n^a, J_m^b]$ of the Kac-Moody algebra (25) to a single integration by introducing a delta function $\delta(p(\zeta_1, \zeta_2, z))$, defined on the already found CP^3 hypersurface (24). Omitting the intermediate calculations, we get the following integral expression for the Kac - Moody algebra (25):

$$\oint d\zeta_2 \tilde{G}(\zeta_2, z) = \tilde{F}(\zeta_2, z) \quad , \quad (44)$$

where $\tilde{F}(\zeta_2, z)$ and $\tilde{G}(\zeta_2, z)$ are the expressions:

$$\tilde{F}(\zeta_2, z) \equiv \frac{1}{2} k n \delta^{ab} \delta_{n+m,0} - \frac{f^{abc}}{2} J^c(\zeta_2) (\zeta_2 - z)^{n+m+3} t^c e^{-i\pi[2(n+m)+3]} \quad , \quad (45)$$

$$\begin{aligned} \tilde{G}(\zeta_2, z) \equiv & \frac{1}{2} J^a(\zeta_1 = z - i(\zeta_2 - z)) J^b(\zeta_2) t^a t^b (\zeta_2 - z)^{m+n+3} \times \\ & \times \left[\varepsilon^{m+1} e^{-\frac{i(m+3)\pi}{2}} - \varepsilon^{n+1} e^{-\frac{i(n+3)\pi}{2}} \right] \quad . \end{aligned} \quad (46)$$

In order to find the "inverse" formulae for the integral representation (44), or in other words - an expression for $\tilde{G}(\zeta_2, z)$ as an integral, depending on the function

$\tilde{F}(\zeta_2, z)$, we shall search for a representation of this function in the form:

$$\tilde{F}(\zeta_2, z) = \int P(X) \delta([X, \eta] - 1) dX \quad . \quad (47)$$

Now it remains to find the function $P(X)$ by making use of the integral inversion formulae (27):

$$P(X) = -\frac{1}{8\pi^2} \tilde{F}(\zeta_2, z) K(X) = -\frac{1}{8\pi^2} \tilde{F}(\zeta_2, z) \int \delta''([X, \eta] - 1) d\eta \quad , \quad (48)$$

then to substitute it into the integral representation formulae (47) and finally, to compare the both sides of (44). The peculiar moment here is that in the L. H. S. of (44) the integration is over the ζ_2 variables, while in the R. H. S. it is over the variables η and $X = (X^0, X^1, X^2, X^3)$. And since X^0 can be expressed through the hyperboloid equation, it remains to integrate over (X^1, X^2, X^3) . In order to be able to compare both sides of (44), we shall pass from an integration over (X^1, X^2, X^3) to an integration over (ζ_2, z) by applying the formulae for the transformation Jacobian:

$$dX^1 \wedge dX^2 \wedge dX^3 \equiv \frac{\partial(X^{i_1}, X^{i_k})}{\partial(\zeta_2, z)} = \sum_{1 \leq i_1 < i_k < 3} \det \begin{pmatrix} \frac{\partial X^{i_1}}{\partial \zeta_2} & \frac{\partial X^{i_1}}{\partial z} \\ \frac{\partial X^{i_k}}{\partial \zeta_2} & \frac{\partial X^{i_k}}{\partial z} \end{pmatrix} d\zeta_2 \wedge dz \quad , \quad (49)$$

where $\frac{\partial X^{i_1}}{\partial \zeta_2}$ simply denotes $\frac{\partial X^{i_1}}{\partial z}$, taken at the point $z = \zeta_2$. The inversion formulae for $\tilde{G}(\zeta_2, z)$ in (44) is obtained in the form

$$\tilde{G}(\zeta_2, z) = -\frac{1}{8\pi^2} \int d\eta dz \tilde{F}(\zeta_2, z) \delta([X, \eta] - 1) \delta''([X, \eta] - 1) \frac{\partial(X^{i_1}, X^{i_k})}{\partial(\zeta_2, z)} N(X) \quad (50)$$

of a complicated nonlinear differential equation in respect to the variables X . The function $N(X)$ comes from the variable change $(X^0, X^1, X^2, X^3) \rightarrow (X^1, X^2, X^3)$, which is calculated by means of the generalization of the Jacobian (49) for a higher dimensional case.

Acknowledgements. The author is grateful to Prof. N. A. Chernikov and Dr. N. S. Shavokhina for the kind invitation to make a report at the International Seminar on Lobachevsky Geometry (26-30 February, 2004, BLTP, JINR, Dubna), dedicated to the 75th Anniversary of Prof. N. A. Chernikov. The author is grateful

also to Prof. I. Todorov, to Prof. S. Manoff and to Dr. N. Nikolov (INRNE, BAS, Sofia, Bulgaria) for useful conversations on CFT and integral geometry and for their interest towards this research. This paper was partially supported also by a Shoumen University grant 005/2002.

References

- [1] N. A. Chernikov, *Lectures on the Lobachevsky Geometry and Relativity Theory*, Novosibirsk, Novosibirsk State University, 1965, in Russian.
- [2] I. M. Gel'fand, M. I. Graev, and N. Y. Vilenkin, *Generalized Functions*, vol.5, Academic Press, New York, 1968.
- [3] I. M. Gel'fand, M. I. Graev, and S. G. Gindikin, *Selected Problems of Integral Geometry*, Dobrosvet, Moscow, 2000.
- [4] M. R. Gaberdiel, An Introduction to Conformal Field Theory, Rept. Progr. Phys. **63** 607-667 (2000) [hep-th/9910156].
- [5] M. R. Gaberdiel, and P. Goddard, Axiomatic Conformal Field Theory, Commun. Math. Phys. **209** 549 - 594 (2000) [hep-th/9810019].
- [6] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite Conformal Symmetry in Two - Dimensional Quantum Field Theory, Nucl. Phys. **B 241** 333 - 380 (1984).
- [7] V. G. Knizhnik, and A. B. Zamolodchikov, Current Algebra and Wess - Zumino Model in Two Dimensions, Nucl. Phys. **B 247** 83 - 103 (1984).
- [8] D. Gepner, and E. Witten, String Theory on Group Manifolds, Nucl. Phys. **B 278** 493 - 549 (1986).
- [9] J. Maldacena, and H. Ooguri, Strings on ADS_3 and the $SL(2, R)$ WZW Model, Part 1: The Spectrum, J. Math. Phys. **42** 2929 - 2960 (2001) [hep-th/0001053].

- [10] J. Maldacena, and H.Ooguri, Strings on ADS_3 and the $SL(2, R)$ WZW Model, Part 2: Euclidean Black Hole, J. Math. Phys. **42** 2961- 2977 (2001) [hep-th/0005183].
- [11] S. Hemming, and E. Keski-Vakkuri, The Spectrum of Strings on BTZ Black Holes and Spectral Flow in the $SL(2, R)$ WZW Model, Nucl. Phys. **B 626** 363 - 376 (2002) [hep-th/0110252].
- [12] A. Giveon, D. Kutasov, and N. Seiberg, Comments on String Theory on ADS_3 , Adv. Theor. Math. Phys. **2** 733-780 (1998) [hep-th/9806194].
- [13] J. Polchinski, *String Theory: Superstring Theory and Beyond*, Cambridge University Press, Cambridge, 1998.
- [14] B. G. Dimitrov, in Proceedings of the First Advanced Research Workshop "Gravity, Astrophysics, and Strings@the Black Sea", Kiten, Bulgaria, June 10-16, 2002, eds.P. Fiziev and M. Todorov, p.64-74, St. Kliment Ohridsky Univ. Press, (2003) [hep-th/0211150].
- [15] I. Bars, Ghost - Free Spectrum of a Quantum String in $SL(2,R)$ Curved Space-time, Phys. Rev. **D53** 3308-3323 (1996) [hep-th/9503205].
- [16] N. V. Yefimov, *Higher Geometry*, 7-th edition, Moscow, Fizmatlit, 2003.
- [17] B. G. Dimitrov, Some Algebro- and Integro- Geometric Aspects of the $SL(2, R)$ Wess-Zumino-Witten Model of Strings on an ADS_3 Background, forthcoming paper.